Lecture 5: Build + Solve the Quantum Scalar Field

Content: Construction of QSF, Fock Space. Improper vectors, normalizations, Heisenberg Picture. Notation.

Building the Quantum Scalar Field:

Klein-Gordon Hamiltonian

$$\hat{H}_{KG} = \frac{1}{2}\int d^3x \left\{ \hat{\pi}^2(x) + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2(x) \right\} \quad (5.1)$$

such that the Equal-Time commutation relation is satisfied

$$[\hat{\phi}(\bar{x}), \hat{\pi}(\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y}), \quad (5.2)$$

which was by the "educated guess" of the classical Klein-Gordon Hamiltonian.

Solution:

$$\hat{H}_{KG} = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\bar{p}}\, \hat{a}^{+}_{\bar{p}}\, \hat{a}_{\bar{p}} \quad (5.3)$$

where $$[\hat{a}_{\bar{p}}, \hat{a}^{+}_{\bar{p}'}] = (2\pi)^3 \delta^{(3)}(\bar{p} - \bar{p}') \tag{5.4}$$

We are not done though!

Why? Since $\omega_{\bar{p}} = \sqrt{|\bar{p}|^2 + m^2}$; we want to call this a <u>relativistic</u> QFT which is <u>not</u> what $\hat{H}_{KG}$ is at the moment.

<u>Construction of Relativistic QFT:</u>

To say we've built a relativistic QFT, we need a projective unitary transformation of the Poincaré group.

Thus far, we have only constructed a <u>subgroup</u> of 1-parameter of the Poincaré group; namely

$$\mathcal{U}((t,0,0,0)) \equiv e^{-i\hat{H}_{KG} t}$$

only time evolution.

We want: Generators for all possible Poincaré group transformations.

We can do this by guessing generators by "putting "acts on" all the conserved charges coming from Noether's theorem $\left(Q_\alpha \longleftrightarrow \hat{Q}_\alpha \right)$

Conservation of currents & charges:

Today: just focus on $p^j \mapsto \hat{p}^j$

generators of transformations

comes from classical mechanics conserved currents $= T^{0j}$.

charges arise from integration over space.

$\boxed{\text{CLASSICAL:}}$

$$T^\mu{}_\nu \overset{KG}{=} \partial^\mu \phi \, \partial_\nu \phi - \delta^\mu_\nu \left(\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \qquad (5.5)$$

conserved currents

Temporal translation

Conserved charge for $x^\mu \longmapsto x^\mu + (a^0, 0, 0, 0)$

$$H = \int T^{00} \, d^3x = \int \mathcal{H} \, d^3x \qquad (5.6)$$

$$\downarrow$$

$$\hat{H} = \int \left(\hat{\ } \quad \hat{\ } \quad \hat{\ } \right) d^3x = \hat{H}_{KG}$$

Already knew!

For spatial transformations,

$$P^j = \int T^{0j}\, d^3x = \int \dot{\phi}\, \partial_j \phi\, d^3x \qquad (5.7)$$

(momentum)

$$= \int \pi\, \partial_j \phi\, d^3x$$

"quantize"

$$\hat{P}^j = \int \hat{\pi}\, \partial_j \hat{\phi}\, d^3x \qquad (5.8)$$

$$\underset{\text{ex. 1}}{=} \int \frac{d^3p}{(2\pi)^3}\, p^j\, \hat{a}_{\vec{p}}^{\dagger}\, \hat{a}_{\vec{p}}$$

Quantization:

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^{\gamma} Q_\gamma \longmapsto [\hat{Q}_\alpha, \hat{Q}_\beta] = i f_{\alpha\beta}{}^{\gamma} \hat{Q}_\gamma \qquad (5.9)$$

? Do $(\hat{H}, \hat{P}^j)$ obey the correct algebra?

$$\{P^\mu, P^\nu\} = 0 \overset{?}{\longmapsto} [\hat{P}^\mu, \hat{P}^\nu] = 0 \qquad (5.10)$$

This implies that we've produced a (proj) unitary group representation of translation subgroup of the Poincaré group!

Now: produce the Hilbert Space.

Hilbert Space Construction (Fock Space):

"Fock Space": Start with "vacuum state" $|\Omega\rangle$ defined by $\hat{a}_{\bar{p}}|\Omega\rangle = 0 \quad \forall \bar{p}$.

Hilbert space is then built as the space generated by all finite linear combinations of vectors of the form

$$|p_1, p_2, \ldots, p_n\rangle = \hat{a}^{\dagger}_{\bar{p}_1} \hat{a}^{\dagger}_{\bar{p}_2} \cdots \hat{a}^{\dagger}_{\bar{p}_n} |\Omega\rangle \qquad (5.11)$$

Completed by the relation $\langle A|A\rangle \equiv |A|^2$.

Problem: the $|p_1, p_2, \ldots, p_n\rangle$ is improper i.e. not normalized. Deal with eventually.

This problem was also encountered in NRQM with the QHO!

Improper Vectors:

"Improper" $\equiv$ "not normalizable" / "innormalizable".

The physical state must be "smeared out" → "continuous" $|p\rangle$ - space

$$|\psi\rangle \equiv \int \frac{d^3p}{(2\pi)^3}\, \psi(p)\, |p\rangle = \int \frac{d^3p}{(2\pi)^3}\, \psi(p)\, \hat{a}^\dagger_{\vec{p}}\, |\Omega\rangle \tag{5.12}$$

where ψ is a smooth (C^2) function.

Another problem: "smearing process" is not obviously Lorentz invariant...

Difficulty comes from "normalization" of $|p\rangle$:

$$\langle p|q\rangle = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}) \tag{5.13}$$

can multiply state by anything - choose so that new state is Lorentz-Invariant. This is called projection.

Improper vector $\xrightarrow[\text{Lorentz Transform}]{\text{normalize}}$ New improper vector.

Goal: Build projector onto single-particle space

$$\mathbb{1}_{\substack{\text{single}\\ \text{particle}}} = \int \frac{d^3p}{(2\pi)^3}\, |p\rangle\langle p| \tag{5.14}$$

not invariant ⟹ not invariant!

This normalizes into SP-state, but isn't invariant.

Introduce some "mystery factor $x(p)$" such that

$$\mathbb{1}_{\text{single-particle}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{x(p)} \, x(p) \, |p\rangle\langle p| \qquad (5.15)$$

and $\displaystyle\int \frac{d^3p}{(2\pi)^3} \frac{1}{x(p)}$ is Lorentz-Invariant.

Claim: $x(p) = \dfrac{2\omega_p}{(2\pi)^3}$.

Proof: $\int d^4p$ is invariant under Poincaré.

↳ 4-volume element.

Consider the Lorentz Transformation

$$(p')^\mu = \Lambda^\mu{}_\nu p^\nu + a^\mu \; ;$$

Jacobian is $\Lambda^\mu{}_\nu$. Note that

$\det(\Lambda) = \pm 1$ ← (An exercise)

And therefore $\int d^4p = \int d^4p'$. → no "stretch factors"

Notice that $p^\mu p_\mu = m^2$ is invariant,

hence $p_0^2 - \omega_p^2 = |p|^2 + m^2$.

"Energy" square

$\omega_p^2 = 0$ since free-particle states

Satisfy $p^\mu p_\mu = m^2$.

Solve for p_0: Two solution branches

$$p_0 = \pm \sqrt{|p|^2 + m^2}$$

"which branch" is Lorentz Invariant ?

So $\displaystyle\int d^4p \; \delta(p_0^2 \underbrace{- |p|^2 - m^2}_{-\omega_p^2})\Big|_{p_0 > 0}$ → $E > 0$ (2.16)

Variational Calculus:

$= \delta((p_0 - \omega_p)(p_0 + \omega_p))$

$= (p_0 + \omega_p)\,\delta(p_0 - \omega_p) + \delta(p_0 - \omega_p)\cancel{(p_0 + \omega_p)}\Big|_{\omega_p = p_0}$

$= 2\omega_p\,\delta(p_0 - \omega_p)$ 0 if $p_0 > 0$

and divide over!

$$= \int \frac{d^3p}{2p_0}\Big|_{p_0 = \omega_p}$$ is invariant.

Thus we define "delta-normalization" for 3-vectors $\vec{v}_a$

$$2\omega_{\bar{p}}\, \delta^{(3)}(\bar{p} - \bar{q}) \qquad (2.17)$$

This implies $\frac{d^3p}{\omega_p} = \frac{d^3p'}{\omega_{p'}}$.

and we build

$$|p\rangle = 2\omega_{\bar{p}} \cdot \hat{a}^\dagger_{\bar{p}} |\Omega\rangle \qquad (2.18)$$

renormalized object

$|\bar{p}\rangle$ is unnormalized, improper

Hence $$\langle q | p \rangle = (2\pi)^3 \cdot 2\omega_{\bar{p}} \cdot \delta^{(3)}(\bar{p} - \bar{q}) \qquad (2.19)$$

which is now an invariant concept.

Action of Renormalized Vectors on Operators:

What is the action of $\hat{p}^{\mu}$ on $|p_1, p_2, \dots, p_n\rangle$?

Lemma: $[\hat{H}_{KG}, \hat{a}_{\bar{p}}] = \omega_{\bar{p}} \hat{a}_{\bar{p}}$; $[\hat{p}^j, \hat{a}_{\bar{p}}] = p^j \hat{a}_{\bar{p}}$. (2.20)

Proof: Exercise.

Corollary: $$\hat{p}^{\mu} |p_1, p_2, \dots, p_n\rangle = \left(\sum_{j=1}^{n} p_j^{\mu}\right) |p_1, \dots, p_n\rangle \quad (2.21)$$

where $p^0 \equiv \omega_{\bar{p}}$ because $\hat{p}^{\mu}|\Omega\rangle = 0$.

Diagonalization of the $\hat{p}$ basis!

$\Longrightarrow$ implement translations quantitatively via

$$\mathcal{U}(a) \equiv e^{-i a_{\mu} \hat{p}^{\mu}} \quad (2.22)$$

Lorentz Invariances in the Heisenberg Picture:

$$O_H = e^{i\hat{H}t} \, O \, e^{-i\hat{H}t}$$

"Interaction picture"

with $$\frac{dO_H}{dt} = i\,[\hat{H}, O_H] \qquad (2.23)$$

In Hess picture, commutation relations become

normally time arguments imply that we work in H. picture.

$$[\hat{\phi}(t,\bar{x}), \hat{\phi}(t,\bar{y})] = 0$$

$\hat{\phi}_H$ $\hat{\pi}_H$

$$= [\hat{\pi}(t,\bar{x}), \hat{\pi}(t,\bar{y})] \qquad (2.24)$$

$$[\hat{\phi}(t,\bar{x}), \hat{\pi}(t,\bar{y})] = i\,\delta^{(3)}(\bar{x}-\bar{y}) \qquad (2.25)$$

But how does $\hat{\phi}$ change with time?

It is easier to describe spatially-localized processes in the Heisenberg picture than fourier-transforming everything in the Schrödinger picture.

We have that

$$\frac{d\hat{\phi}(t,\bar{x})}{dt} = i\,[\hat{H}, \hat{\phi}(t,\bar{x})]$$

$$= \frac{i}{2}\Big[\int d^3y\ \hat{\pi}^2(t,\bar{y}) + \underline{(\nabla\hat{\phi}(t,\bar{y}))^2} + \underline{m^2\hat{\phi}^2(t,\bar{y})}, \hat{\phi}(t,\bar{x})\Big]$$

vanishes by (2.24). $\qquad (2.26)$

excercise!

$$= \hat{\pi}(t,\bar{x}) \qquad (2.27)$$

Alike Hamilton's equations,

$$\frac{d\hat{\pi}(t,\bar{x})}{dt} = i\,[\hat{H},\ \hat{\pi}(t,\bar{x})]$$

$$= \frac{i}{2}\Big[\int d^3y\ \hat{\pi}^2(t,\bar{y}) + (\nabla\hat{\phi}(t,\bar{y}))^2 + m^2\hat{\phi}^2(t,\bar{y})\ ,\ \hat{\pi}(t,\bar{x})\Big] \qquad (2.28)$$

$$= \frac{i}{2}\int d^3y\ \Big\{ 2i\ (\nabla_j\hat{\phi}(t,\bar{y}))(\nabla_j\delta^{(3)}(\bar{x}-\bar{y})) + 2m^2\cdot i\ \delta(\bar{x}-\bar{y})\ \hat{\phi}(t,\bar{y}) \Big\} \qquad (2.29)$$

$$= \nabla^2\hat{\phi}(t,\bar{x}) - m^2\hat{\phi}(t,\bar{x}) \qquad (2.30)$$

But <u>this</u> implies that

$$\boxed{\partial_\mu\partial^\mu\ \hat{\phi}(t,\bar{x}) + m^2\hat{\phi}(t,\bar{x}) = 0} \qquad (2.31)$$

i.e. the field operator $\hat{\phi}$ obeys the Klein-Gordon equation! Good sign.

Excercises :

1. Prove that $\hat{p}^j = \int \hat{\pi}\, \partial_j \hat{\phi}\, d^3x = \int \frac{d^3p}{(2\pi)^3}\, p^j\, \hat{a}^\dagger_{\bar{p}}\, \hat{a}_{\bar{p}}$.

Proof: First equatorial is given by energy-momentum tensor T^{0j} components integrated over $\mathbb{R}^3$.

Second equatorial given by noting that

$$\hat{\phi} =$$

$$\det(\Lambda) = \pm 1 \leftarrow \text{(An excercise)}$$

Lemma: $[\hat{H}_{KG}, \hat{a}_{\bar{p}}] = \omega_{\bar{p}}\, \hat{a}_{\bar{p}}$; $[\hat{p}^j, \hat{a}_{\bar{p}}] = p^j\, \hat{a}_{\bar{p}}$. (2.20)

Proof: Excercises.

$$\frac{d\phi(t,\bar{x})}{dt} = i\left[\hat{H}, \hat{\phi}(t,\bar{x})\right]$$

$$= \frac{i}{2}\left[\int d^3y \; \hat{\pi}^2(t,\bar{y}) + (\nabla\underline{\hat{\phi}(t,\bar{y})})^2 + \underline{m^2\hat{\phi}^2}(t,\bar{y}) \;,\; \hat{\phi}(t,\bar{x})\right]$$

Vanishing by (2.24). (2.26)

excercise!

$$\uparrow$$

$$= \hat{\pi}(t,\bar{x})$$